Towards a Propositional Logical Structure of Ambiguous Words in Weighted Automata

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Workshop on Ambiguity 2018

- Non-deterministic Finite (Weighted) Automata;
- Degrees of Ambiguity in Finite Automata;
  - (Ravikumar and Ibarra, 1989), (Leung, 1998), (Leung, 2005);
- Weighted Automata vs Logics;
  - (Droste and Gastin, 2007), (Gerla, 2003-2004), (Schwartz, 2006);
- Łukasiewicz logic;
  - Interpreting Paths as Functions.

A **Non-deterministic Finite Automaton** (NFA) is a tuple $\mathcal{A} = \langle \Sigma, S, I, F, \delta \rangle$, where:

- Σ is a finite alphabet;
- S is a finite set of **states**;
- $I \subseteq S$ is a set of *initial states*;
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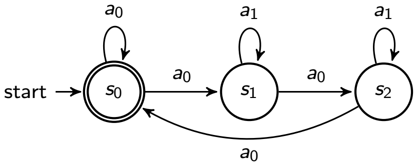
- a **path** is a finite sequence of states $s_1, \dots, s_m \in S$;
- a **word** is a finite sequence $w = a_1 \cdot a_2, \cdots a_n$ of **length** n, of **letters** a_i from Σ;
- a **run** of a NFA over a word w is a path $s_1, \dots, s_n$, such that $s_1 \in I$ and $s_{i+1} \in \delta(s_i, a_i)$ for any $1 \leq i \leq n$;
- a word w is **accepted** if the a run $s_1, \dots, s_n$ is such that $s_n \in F$;

The **language recognized** by $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of words accepted by $\mathcal{A}$.

Every NFA $\mathcal{A}$ can be tranformed in a *deterministic FA* accepting the same language:



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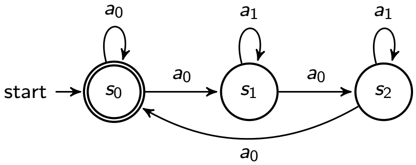

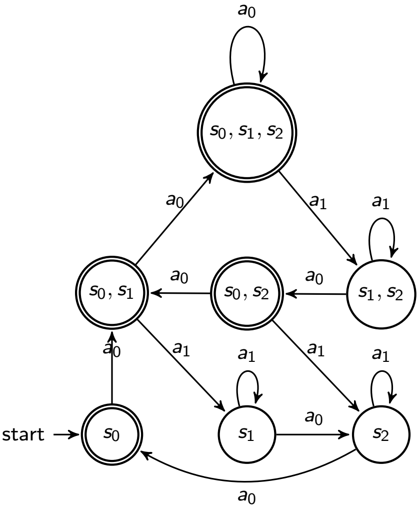

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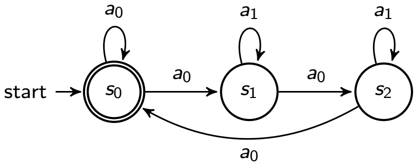
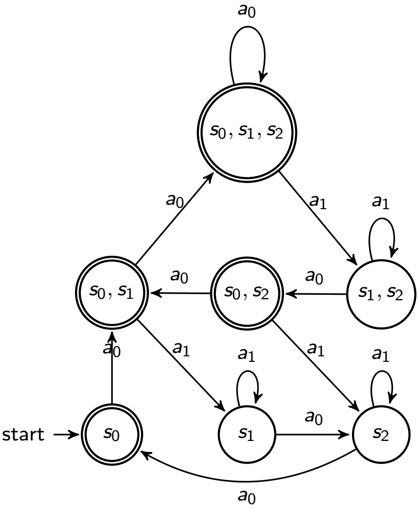


In the worst case, from a NFA with n states, we build a DFA with $2^n - 1$ states (Leung. 1998).



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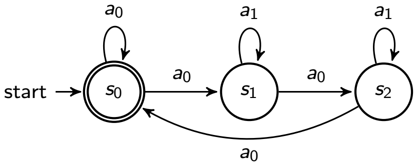

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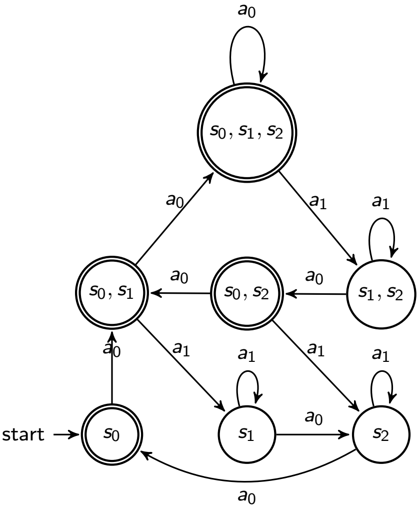
The complexity of such operation is tied to the **degree of ambiguity** of $\mathcal{A}$.



The **degree of ambiguity** d_w of a word w is the number of different accepting paths for w in $\mathcal{A}$.

$d_{\mathcal{A}}(n)$ is the maximum of the degrees of ambiguity of words of length n or less,

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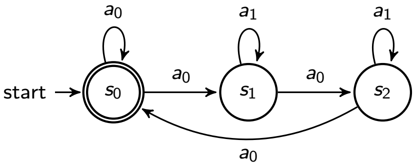

$\lvert w \rvert$	w		paths
3	$a_0a_0a_0$	$d_w = 2$	↷↷↷, →→→
4	$a_0a_0a_0a_0$	$d_w = 3$	↷↷↷↷, →→→↷, ↷→→→
5	$a_0a_0a_0a_0a_0$	$d_w = 4$	↷↷↷↷↷, →→→↷↷, ↷↷→→→, ↷→→→↷
6	$a_0a_0a_0a_0a_0a_0$	$d_w = 6$	→→→→→→, ↷↷↷↷↷↷, →→→↷↷↷, ↷↷↷→→→, ↷↷→→→↷, ↷→→→↷↷

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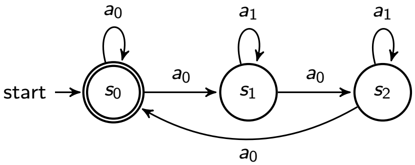

$\lvert w \rvert$	w		paths
3	$a_0a_0a_0$	$d_w = 2$	$\curvearrowright\curvearrowright\curvearrowright, \rightarrow\rightarrow\rightarrow$
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5	$a_0a_0a_0a_0a_0$	$d_w = 4$	$\curvearrowright\curvearrowright\curvearrowright\curvearrowright\curvearrowright, \rightarrow\rightarrow\rightarrow\curvearrowright\curvearrowright, \curvearrowright\curvearrowright\rightarrow\rightarrow\rightarrow, \curvearrowright\rightarrow\rightarrow\rightarrow\curvearrowright$
6	$a_0a_0a_0a_0a_0a_0$	$d_w = 6$	$\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow, \curvearrowright\curvearrowright\curvearrowright\curvearrowright\curvearrowright\curvearrowright, \rightarrow\rightarrow\rightarrow\curvearrowright\curvearrowright\curvearrowright,$ $\curvearrowright\curvearrowright\curvearrowright\rightarrow\rightarrow\rightarrow, \curvearrowright\curvearrowright\rightarrow\rightarrow\rightarrow\curvearrowright, \curvearrowright\rightarrow\rightarrow\rightarrow\curvearrowright\curvearrowright$

$d_{\mathcal{A}}(|w|) >= 2^{\lfloor |w|/3 \rfloor}$ (Leung, 1998)

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- UFA $\mathcal{A}$ is **unambiguous** if $d(\mathcal{A}) \leq 1$;
- FNA $\mathcal{A}$ is **finitely ambiguous** if $d(\mathcal{A}) \leq c$, where c is a constant function;
- PNA $\mathcal{A}$ is **polinomially ambiguous** if $d(\mathcal{A}) \leq p$, where p is a polynomial function;
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Separating NFA

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$$DFA <_p UFA <_p FNA \leq_p PNA <_p NFA$$

where $C \leq_p C'$ when there exists a polynomial p such that for any finite automaton in C with n states, it is possibile to find an equivalent automaton in C' with $p(n)$ states.

References: (Ravikumar and Ibarra, 1989), (Leung, 2005);

Ambiguity Matters!

Ambiguity is relatead to the succinctness in the number of states.

Restricting Ambiguity $\longrightarrow$ Increases the number of states.

Ambiguity influences the tractability of algorithmic issues.

For instance, for every $n \in \mathbb{N}$ it can be determined efficiently if two NFA of ambiguity at most n are equivalent.

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- $e : S \times \Sigma \times S \to \mathbf{R}$ is a **weight** function, assigning to each triple $(s, a, s') \in S \times \Sigma \times S$ a value $r \in \mathbf{R}$.

A **semiring** is a structure $\mathbf{R} = \langle R, +, \cdot, 0, 1 \rangle$ such that:

- $\langle R, +, 0 \rangle$ is a commutative monoid, and $\langle R, \cdot, 1 \rangle$ is a monoid;
- $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for $x, y, z \in R$;
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Let $p = s_1, \ldots, s_n$ be an accepting path for a word $w = a_1 \ldots a_n$.

The **weight** of p is

$$\|p\| = \prod_{1 \leq i \leq n-1} e(s_i, a_i, s_{i+1}).$$

The **weight** of a word w is

$$\|w\| = \sum_{p \in P_w} \|p\|.$$

The **behavior** of $\mathcal{A}$ is a map $\|\mathcal{A}\| : \Sigma^* \to \mathbf{R}$ that sends every $w \in \Sigma^*$ to $\|w\|$.

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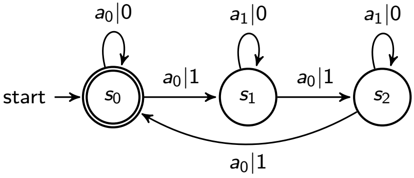
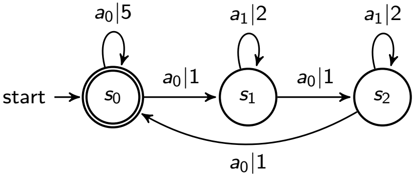
An **R-valued language** is a map $L : \Sigma^* \to \mathbf{R}$.
An **R**-valued language is **recognizable** if there exists a NWA $\mathcal{A}$ s.t. $\|\mathcal{A}\| = L$.



$\mathbf{B} = \langle \{0, 1\}, \vee, \wedge, 0, 1 \rangle$



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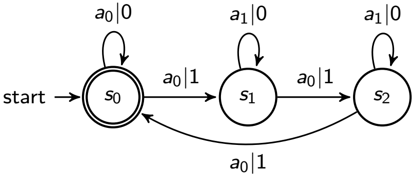

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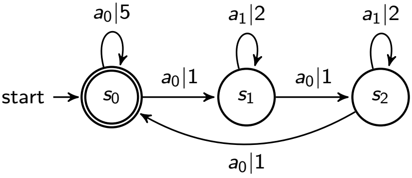

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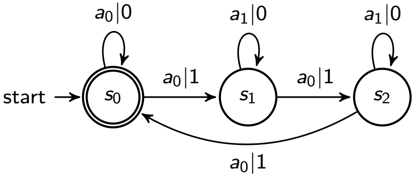
For every $x, y \in [0,1]$ define:

$$x \oplus y = min\{1, x + y\},$$
$$x \odot y = max\{0, x + y - 1\},$$

both $\oplus$ and $\odot$ are commutative.



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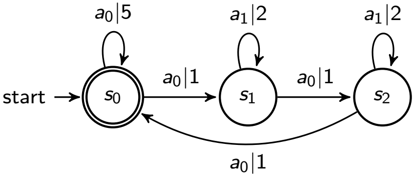

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Then, the structures

$$[0,1]^{\wedge} = \langle [0,1], \wedge, \oplus, 0, 1 \rangle,$$
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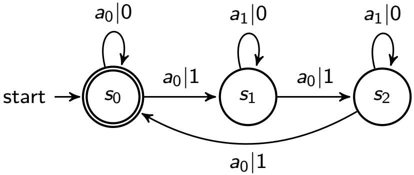
are two commutative semirings.

Weighted Automata



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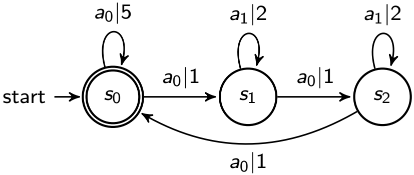

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Definitions of degree of ambiguity of a word d_w and of automata $d_{\mathcal{A}}(n)$ are the same as for NFA.

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An evaluation is a function $\mu : \textsc{Form} \to [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \textsc{Form}$,

$$\mu(\bot) = 0, \qquad \mu(\top) = 1, \qquad \mu(\neg\alpha) = 1 - \mu(\alpha),$$
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Schwartz's results are based on the relation between MV-algebras and semirings, and the definition of MV-automata defined in the works of B. Gerla (2003-2004).

A **MV-algebra** is a structure $\mathbf{A} = \langle A, \oplus, \odot, \neg, \bot, \top \rangle$ such that:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z \quad x \oplus y = y \oplus x, \quad x \oplus \bot = x, \quad x \oplus \top = \top,$$

$$\neg\bot = \top, \quad \neg\neg x = x,$$

$$\neg(\neg x \oplus \neg y) = x \odot y, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

In a MV-algebra **A** we have:

$$x \leq y \text{ iff } \neg x \oplus y = \top, \quad x \vee y = x \oplus (\neg x \odot y), \quad x \wedge y = x \odot (\neg x \oplus y).$$

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Theorem (Gerla, 2003)

For every MV-algebra $\mathbf{A}$ *both structures:*

$$\mathbf{A}^{\vee} = \langle A, \vee, \odot, \bot, \top \rangle \text{ and } \mathbf{A}^{\wedge} = \langle A, \wedge, \oplus, \bot, \top \rangle,$$

are commutative semirings,
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$$1 — \tfrac{5}{6} — \tfrac{4}{6} — \tfrac{3}{6} — \tfrac{2}{6} — \tfrac{1}{6} — 0$$

$$\neg\left(\frac{1}{6} \vee \frac{4}{6}\right) = \neg\frac{1}{6} \wedge \neg\frac{4}{6}$$

$$\neg\left(\frac{3}{6} \odot \frac{4}{6}\right) = \neg\frac{3}{6} \oplus \neg\frac{4}{6}$$

An **MV-Automaton** is an NWA $\mathcal{A} = \langle \Sigma, S, I, F, \delta, e \rangle$ where e takes values over an MV-algebra $\mathbf{A} = \langle A, \oplus, \odot, \neg, \bot, \top \rangle$.

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Let $p = s_1, \ldots, s_n$ be an accepting path for a word $w = a_1 \ldots a_n$.

The **weight** of p is

$$\|p\| = \bigodot_{1 \leq i \leq n-1} e(s_i, a_i, s_{i+1}).$$

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$$\|w\| = \bigvee_{p \in P_w} \|p\|.$$

The **behavior** of $\mathcal{A}$ is a map $\|\mathcal{A}\| : \Sigma^* \to \mathbf{A}$ that sends every $w \in \Sigma^*$ to $\|w\|$.

An **A-valued language** is a map $L : \Sigma^* \to \mathbf{A}$.
An **A**-valued language is **recognizable** if there exists a MVA $\mathcal{A}$ s.t. $\|\mathcal{A}\| = L$.

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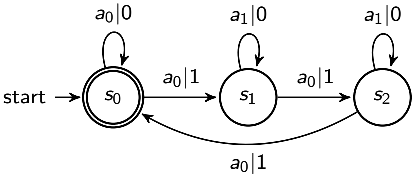
The class of **A**-valued languages $\mathbf{A}^{\Sigma^*}$, can be equipped by pointwise MV-algebras operations (Gerla, 2004).

Theorem (Gerla, 2004)

$$\|\mathcal{A} \vee \mathcal{B}\| = \|\mathcal{A}\| \vee \|\mathcal{B}\| \qquad \|\mathcal{A} \odot \mathcal{B}\| = \|\mathcal{A}\| \odot \|\mathcal{B}\|$$

Take the following NWA over
$\mathbf{B} = \langle \{0,1\}, \vee, \wedge, 0, 1 \rangle$.



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Accepting paths

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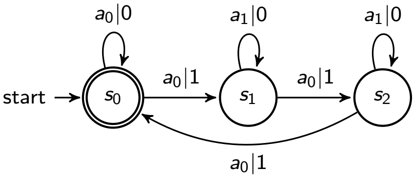

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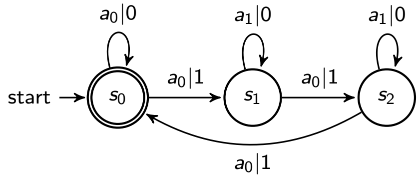

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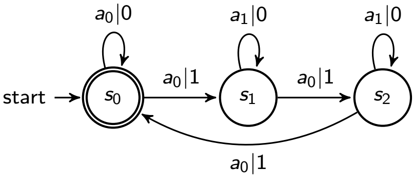

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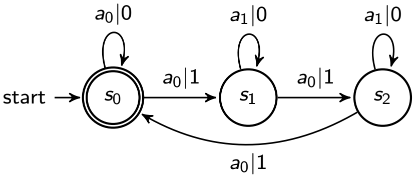

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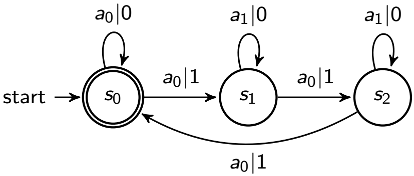

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Cardinality issues?
The number of Boolean assignments of a formula in n variable is 2^n.

Let P_w be the set of accepting paths of $w = a_1, \ldots, a_n$.
When $|P_w| > 2^{|w|}$, then there exist $(a_1, \ldots, a_n), (a_1, \ldots, a_n)' \in P_w$ s.t. $(\|a_1\|, \ldots, \|a_n\|) = (\|a_1\|, \ldots, \|a_n\|)'$.

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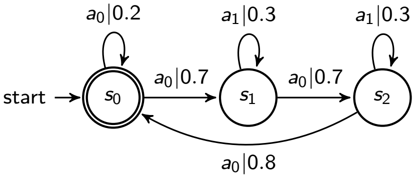

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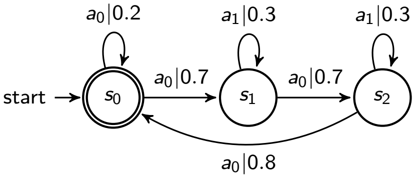

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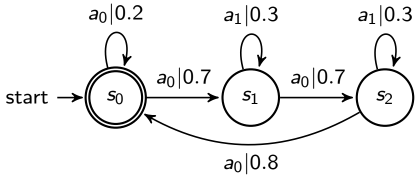

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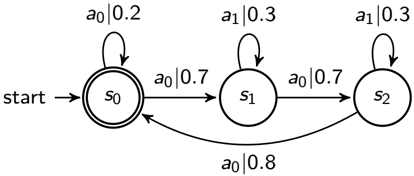

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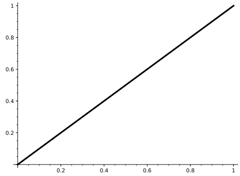

Semiring over the Real Unitary Interval

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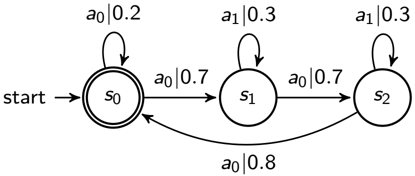

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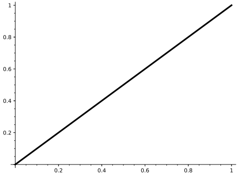
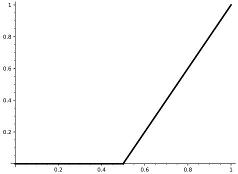
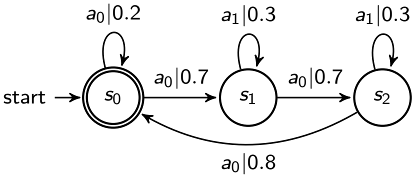


$r_1 \odot r_1$



Take the following NWA over $[0,1]^{\vee} = \langle [0,1], \vee, \odot, 0, 1 \rangle$.



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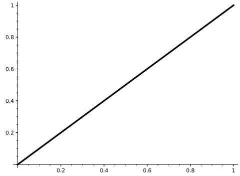

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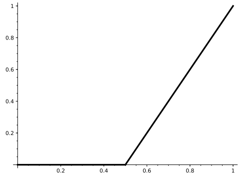

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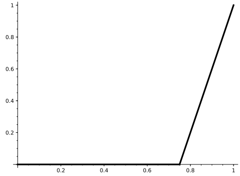

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McNaughton functions

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- there are affine linear polynomials $f_1, \dots f_k$ with integer coefficients, $f_i(x_1, \dots, x_n) = b_i + a_{i1}x_1 + \dots + a_{in}x_n$, such that for each point $\mathbf{p} = (p_1 \dots, p_n) \in [0,1]$ there is a $j \in \{1, \dots, k\}$ with $f(\mathbf{p}) = f_j(\mathbf{p})$.

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From every word w in $[0,1]^\vee = \langle [0,1], \vee, \odot, 0, 1 \rangle$,
we obtain a function $f_w : ([0,1]^\vee)^n \to [0,1]^\vee$
such that there exist a corresponding propositional formula φ_{f_w} in Łukasiewicz logic.

Notice that this is valid also in the case of $[0,1]^\wedge$ or over a MVA.

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$F_1 = \{0, 1\}$, $F_2 = \{0, \frac{1}{2}, 1\}$, $F_3 = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$, ...
for any $\frac{a_1}{d_1}, \frac{a_2}{d_2}$ in F_i, we add their *Farey's mediant* $\frac{a_1+a_2}{d_1+d_2}$ to F_{i+1}.

McNaughton functions

The full construnction from McNaughton functions to formulas in Ł is lenghty. Here, we briefly sketch a useful decomposition property of McNaughton functions, that shed a light on how to obtain logical formulas.

$\mathrm{F}_1 = \{0, 1\}$, $\mathrm{F}_2 = \{0, \frac{1}{2}, 1\}$, $\mathrm{F}_3 = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$, ...
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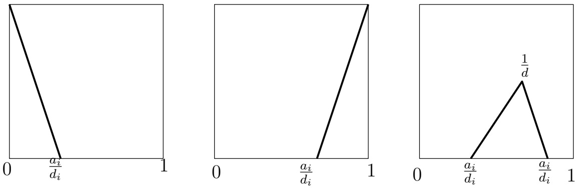
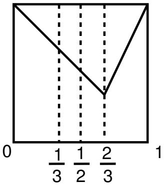
The Farey series help us to obtain a *unimodular* partition $\mathcal{U}$ of $[0,1]^n$ such that for a vertex $\{\mathbf{v}\} \in \mathcal{U}$, a *Schauder hat* is the unique continuous function $h_{\mathbf{v}}$ s.t.:

- $h_{\mathbf{v}} = 1/\mathrm{den}\,(\mathbf{v})$, and it is linear on every simplex of $\mathcal{U}$.
- $h_{\mathbf{x}} = 0$ for every simplex S in $\mathcal{U}$ such that $\mathbf{v} \in S$.



McNaughton functions

$$f(x) = \begin{cases} 1 - x & x \in [0, \frac{2}{3}) \\ 2x - 1 & x \in [\frac{2}{3}, 1] \end{cases}$$



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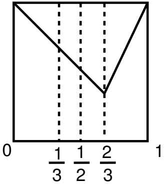

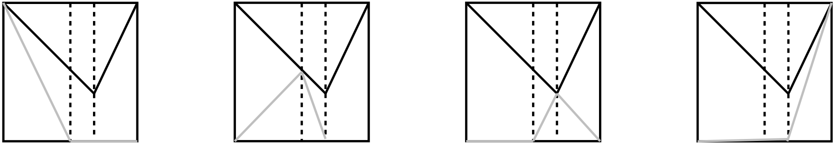

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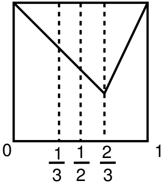

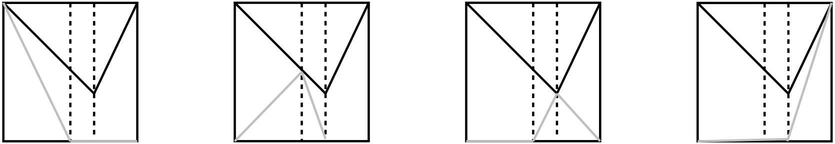

We can express every McNaughton function as a sum of Schauder hats $h_{\mathbf{v}}$:

$$f = \bigoplus k_{\mathbf{v}} h_{\mathbf{v}}$$

where $f(\mathbf{v}) = k_{\mathbf{v}}/(den\mathbf{v})$.

Formulas

To each Schauder hat $h_{\mathbf{v}}$ it is possibile to associate a Ł formula $\varphi_{h_{\mathbf{v}}}$ that takes the role of a **minterm** in Ł.

That is, every φ in Ł can be constructed as a $\oplus$-disjunction *of minterms* $\bigoplus_{\mathbf{v} \in \mathcal{U}} m_{\mathbf{v}} \cdot \varphi_{h_{\mathbf{v}}}$.

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Let p be an accepting path for a word in $[0,1]^{\vee} = \langle [0,1], \vee, \odot, 0, 1\rangle$,
instead of taking a function $f_p : ([0,1]^{\vee})^n \to [0,1]^{\vee}$ as before,
we just take a point $\mathbf{v} \in ([0,1]^{\vee})^n$ such that $\mathbf{v} = (\|a_1\|, \ldots, \|a_n\|)$.

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Notice that, this is quite different than in the classical case $\{0,1\}$.

Indeed, the number of paths/assignments over $\{0,1\}^n$ is finite 2^n,
while in the semirings over $[0,1]$ we have infinite assignments, allowing us to associate different Ł formulas to different accepting paths.

A *commutative integral bounded residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that
$(A, \wedge, \vee, \bot, \top)$ is a bounded lattice,
$(A, \odot, \top)$ is a commutative monoid,
and the *residuation* equivalence, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, holds.

An **MTL algebra** $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ is a commutative integral bounded residuated lattice satisfying the **prelinearity** equation,

$$(x \rightarrow y) \vee (y \rightarrow x) = \top$$

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Every MTL algebra $\mathbf{A}$ admits a semiring reduct $\mathbf{S}^{\vee} = \langle A, \vee, \odot, \bot, \top \rangle$.

To generalize/apply Gerla's approach to other MTL algebras, they need also $\mathbf{S}^{\wedge} = \langle A, \wedge, \oplus, \bot, \top \rangle$.

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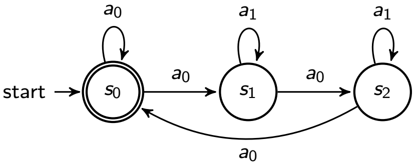
But $\oplus$ is not always nicely definable (Aguzzoli, Bianchi, Flaminio. 2015).

One interesting case, where $\oplus$ is definable is the variety of algebras corresponding to NM logic, that is *Constructive Logic with Strong Negation* plus prelinearity.

Another interesting (dual) approach to deal with DFA in propositional logic is the one of (Aguzzoli, Diaconescu, Flaminio. 2014).



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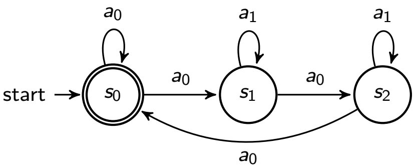

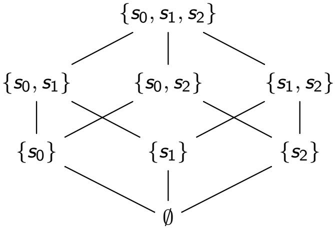

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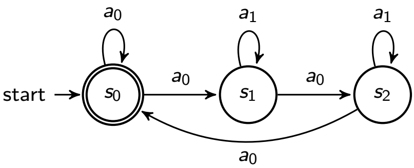

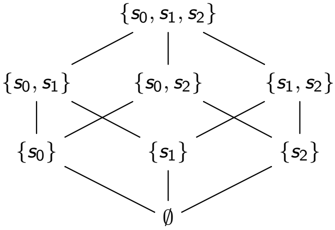

Can we generalize this approach to NFA, NWA $[0,1]^{\wedge} = \langle [0,1], \wedge, \oplus, 0, 1 \rangle$, or to MVA?

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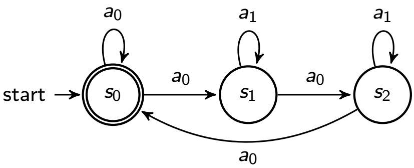
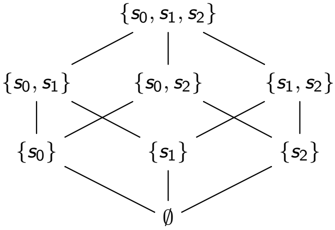
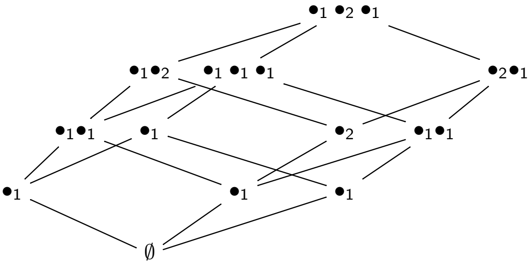




Can we generalize this approach to NFA, NWA $[0,1]^\wedge = \langle [0,1], \wedge, \oplus, 0, 1\rangle$, or to MVA?

(Finite) Duality Theory!

(for $\mathbb{MV}_3$ in the picture on the right)



- Interpreting paths as assignments, we have introduced an additional *level* of ambiguity (different paths may produce the same evaluation);
  - What does it mean? (Disambiguation?)
- We linked words over NWA to propositional (many-valued) logics formulas;
  - Does words inherits the "logical structure" of the Lindenbaum algebra?
  - Is ambiguity on words reflected in the formulas?
  - Can logics tell us something about ambiguous NFA separation?

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